Journal of Pure and Applied Mathematics: Advances and Applications
Volume 11, Number 2, 2014, Pages 137-146

ON THE NONASSOCIATIVE JEWELL-SINCLAIR THEOREM

AMIR A. MOHAMMED and SUHAM M. ALI

Department of Mathematics
College of Education
University of Mousul
Mousul
Iraq

Department of Mathematics
College of Sciences
University of Kirkuk
Kirkuk
Iraq
e-mail: sumoal_73@yahoo.com

Abstract

A g-c-derivation is a linear mapping D from a normed algebra $\mathcal{A}$ into itself such that $D(ab) = D(a)g(b) + aD(b) = D(a)b + g(a)D(b)$ for all $a, b \in \mathcal{A}$, where g is continuous linear map from $\mathcal{A}$ into itself. In this paper, we prove that any g-c-derivation on a semiprime Banach nonassociative algebra $\mathcal{A}$ is continuous if for each closed infinite dimensional ideal $I \subseteq \mathcal{A}$, there is a sequence $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ (the multiplication algebra of $\mathcal{A}$), such that the sequence $\{(T_1 T_2 \cdots T_n I)^-\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing. As a consequence, every g-c-derivation on nonassociative H^*-algebra with zero annihilator is continuous.

2010 Mathematics Subject Classification: 17Dxx, 13N15, 46H40.
Keywords and phrases: nonassociative algebras, derivations, automatic continuity.
Received March 7, 2014

1. Introduction

Jewell and Sinclair in [5] obtained the continuity of derivations on certain Banach algebras known as Jewell-Sinclair theorem. In 1994, Palmer in [8] presented Jewell-Sinclair theorem in a perfect form as follows: Every derivation on a Banach algebra $\mathcal{A}$ is continuous if $\mathcal{A}$ satisfies:

(i) $\mathcal{A}$ has no nonzero finite dimensional nilpotent ideals.

(ii) For each closed infinite dimensional ideal I of $\mathcal{A}$, there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that the sequence $\{(a_1 a_2 \cdots a_n I)^-\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing.

In [6], they generalized the above Palmer presentation of Jewell-Sinclair theorem in the nonassociative setting as follows: Let $\mathcal{A}$ be a semiprime Banach algebra not necessarily associative, such that for each closed infinite dimensional ideal $I \subseteq \mathcal{A}$, there is a sequence $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ (the multiplication algebra of $\mathcal{A}$), such that the sequence $\{(T_1 T_2 \cdots T_n I)^-\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing, then any derivation on $\mathcal{A}$ is continuous. Also, they used this result and Villena's lines proof in [11, Theorem 4] to prove that every derivation on nonassociative H^*-algebra with zero annihilator is continuous.

In this paper, we will prove that every g-c-derivation on nonassociative H^*-algebra with zero annihilator is continuous via nonassociative Jewell-Sinclair theorem. So our purpose is the following theorems:

Theorem A. *Let $\mathcal{A}$ be a semiprime complete normed algebra such that for each closed infinite dimensional ideal $I \subseteq \mathcal{A}$, there is a sequence $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ such that the sequence $\{(T_1 T_2 \cdots T_n I)^-\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing. Then any g-c-derivation on $\mathcal{A}$ is continuous.*

Theorem B. *Every g-c-derivation on nonassociative H^*-algebra $\mathcal{A}$ with zero annihilator is continuous.*

Following [7], we define a g-c-derivation as follows : Let $\mathcal{A}$ be a normed algebra, a linear mapping D from $\mathcal{A}$ into itself is called g-c-derivation, if $D(ab) = D(a)g(b) + aD(b) = D(a)b + g(a)D(b)$ for all $a, b \in \mathcal{A}$, where g is continuous linear map from $\mathcal{A}$ into itself, note that if g is the identity map, then D is the usual derivation. Recall from [10] that, if X and Y are normed spaces and if $T : X \to Y$ is linear map, then *separating subspace* $\mathcal{S}(T)$ of T is define as follows: $\mathcal{S}(T) = \{y \in Y : \exists$ a sequence $\{x_n\}$ in $X, n \in \mathbb{N}$ with $\lim x_n = 0$ and $\lim T(x_n) = y\}$. The separating space $\mathcal{S}(T)$ is a closed linear subspace of Y. Also, recall from [1] that an *annihilator* of an algebra $\mathcal{A}$ (denoted by $\operatorname{Ann}(\mathcal{A})$) is defined as the set of those a in $\mathcal{A}$ satisfying $ab = ba = 0$ for every $b \in \mathcal{A}$. An algebra $\mathcal{A}$ is *semiprime* if for any ideal I of $\mathcal{A}$ such that $I^2 = 0$, then $I = 0$ and $\mathcal{A}$ is *prime*, if for any two ideals I and J of $\mathcal{A}$ such that $I.\, J = 0$, then either $I = 0$ or $J = 0$. Also, if $\mathcal{A}$ has nonzero product and has no nonzero proper closed ideals, then $\mathcal{A}$ is *topologically simple*. The *multiplication algebra* of $\mathcal{A}$ denoted by $M(\mathcal{A})$ is defined as a subalgebra of $L(\mathcal{A})$ (the algebra of all linear mapping on $\mathcal{A}$) generated by L_a, R_a, and $Id_{\mathcal{A}}$, which is left, right, and identity multiplication operators, respectively. An *involution* of an algebra $\mathcal{A}$ is a mapping $x \to x^*$ of $\mathcal{A}$ into $\mathcal{A}$ such that for all $x, y \in \mathcal{A}, \alpha \in \mathbb{C}$ (complex field) the mapping* satisfies the following conditions:

(i) $(x + y)^* = x^* + y^*$; (ii) $(\alpha x)^* = \alpha^* x^*$; (iii) $(x^*)^* = x$; (iv) $(xy)^* = y^* x^*$.

A nonassociative *H^*-algebra* is an algebra $\mathcal{A}$ with algebra involution*, whose underlying vector space is a Hilbert space satisfying $\langle ab, c\rangle = \langle a, cb^*\rangle = \langle b, a^*c\rangle$.

Finally, we recall that a prime algebra $\mathcal{A}$ over a field $\mathbb{C}$ is said to be *centrally closed*, if for every nonzero ideal I of $\mathcal{A}$ and for every linear mapping $f : I \to \mathcal{A}$ with $f(ax) = af(x)$ and $f(xa) = f(x)a$, for all $a \in \mathcal{A}$ and $x \in I$, then there exists $\lambda \in \mathbb{C}$ such that $f(x) = \lambda x$ for all $x \in I$ (see [4]). From now on, in this paper, all algebra are not necessarily associative over a complex field.

2. Proof of Theorem A

For a Banach space X, we denote by $BL(X)$ (the Banach space of all bounded linear mapping on X). We begin this section by the following results:

Lemma 2.1 ([8, Lemma 6.1.17]). *Let X and Y be Banach spaces. Let $\{S_n\}_{n\in\mathbb{N}} \subseteq BL(X)$ and $\{R_n\}_{n\in\mathbb{N}} \subseteq BL(Y)$ and $T \in L(X, Y)$ satisfy $TS_n - R_nT \in BL(X, Y)$ for all $n \in \mathbb{N}$. Then, there is an integer k such that $(R_1R_2 \cdots R_n\mathcal{S}(T))^- = (R_1R_2 \cdots R_k\mathcal{S}(T))^-$ for all $n \geq k$.*

Lemma 2.2. *If $\mathcal{A}$ is a normed algebra and if D is g-c-derivation on $\mathcal{A}$. Then $\mathcal{S}(D)$ is closed ideal of $\mathcal{A}$.*

Proof. It is clear that $\mathcal{S}(D)$ is a closed subspace of $\mathcal{A}$. Let $b \in \mathcal{S}(D)$, there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that $\lim a_n = 0$ and $\lim D(a_n) = b$. For all $a \in \mathcal{A}$, we have $\lim a_na = 0$. Since g is continuous, it follows that $\lim g(a_n) = 0$. Now $\lim D(a_na) = \lim D(a_n)a + \lim g(a_n)D(a) = ba$. Therefore $ba \in \mathcal{S}(D)$. Similarly, $ab \in \mathcal{S}(D)$. This complete the proof. □

Lemma 2.3. *If $\mathcal{A}$ is a normed algebra and if D is g-c-derivation on $\mathcal{A}$, then $DT - TD \in BL(\mathcal{A})$, for all $T \in M(\mathcal{A})$.*

Proof. Let $\mathcal{F} = \{T \in BL(\mathcal{A}) : DT - TD \in BL(\mathcal{A})\}$. It is clear that $\mathcal{F}$ is subspace of $BL(\mathcal{A})$. For $T_1, T_2 \in \mathcal{F}$, we have $DT_1T_2 - T_1T_2D = (DT_1 - T_1D)T_2 + T_1(DT_2 - T_2D)$. Therefore $DT_1T_2 - T_1T_2D \in BL(\mathcal{A})$, that is, $T_1T_2 \in \mathcal{F}$. As a consequence, $\mathcal{F}$ is a subalgebra of $BL(\mathcal{A})$. Since L_a, R_a, and $Id_{\mathcal{A}}$ are in $\mathcal{F}$, it follows that $\mathcal{F} = M(\mathcal{A})$. □

The proof of the following theorem is similar to that given in [6, Theorem 2-1].

Theorem A. *Let $\mathcal{A}$ be a semiprime complete normed algebra such that for each closed infinite dimensional ideal $I \subseteq \mathcal{A}$, there is a sequence $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ such that the sequence $\{(T_1T_2 \cdots T_nI)^-\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ is constantly decreasing. Then any g-c-derivation on $\mathcal{A}$ is continuous.*

Proof. Let D be a g-c-derivation on $\mathcal{A}$. By Lemma 2.2, $\mathcal{S}(D)$ is closed ideal of $\mathcal{A}$. If $\mathcal{S}(D)$ is infinite dimensional, then by assumption, there is a sequence $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ such that the sequence $\{(T_1T_2 \cdots T_n\mathcal{S}(D))^-\}_{n\in\mathbb{N}}$ is constantly decreasing. Applying Lemmas 2.1 and 2.3 by setting: $X = Y = \mathcal{A}$, $T = D$, $R_n = T_n = S_n$ we get, there exist a natural number $k \in \mathbb{N}$ such that $(T_1T_2 \cdots T_n\mathcal{S}(D))^- = (T_1T_2 \cdots T_k\mathcal{S}(D))^-$ for all $n \geq k$. This condition implies that $\{T_n\}_{n\in\mathbb{N}}$ not constantly decreasing, a contradiction. So, $\mathcal{S}(D)$ must be finite dimensional. Note that, if $\mathcal{S}(D)$ is finite dimensional, then $D|_{\mathcal{S}(D)}$ is continuous. Now we claim that $\mathcal{S}(D)^2 = \{0\}$. Let $a, b \in \mathcal{S}(D)$, then there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that $\lim a_n = 0$ and $\lim D(a_n) = a$. Now since $\lim a_n = 0$, then $\lim a_nb = 0$ and since g is continuous, then $\lim g(a_n) = 0$. Since $\mathcal{S}(D)$ is an ideal of

$\mathcal{A}$ and $b \in \mathcal{S}(D)$, then $a_n b \in \mathcal{S}(D)$ but $D|_{\mathcal{S}(D)}$ is continuous, then $\lim D(a_n b) = 0$. Since $D(a_n b) = D(a_n)b + g(a_n)D(b)$, then $\lim D(a_n b) = ab$, this implies that $ab = 0$. Therefore $\mathcal{S}(D)^2 = \{0\}$ and since $\mathcal{A}$ is semiprime, we have $\mathcal{S}(D) = 0$ and by closed-graph theorem, we obtain that D is continuous. As required. □

3. Proof of Theorem B

In this section, we need the following lemmas before we give our proof of Theorem B:

Lemma 3.1 ([11]). *Let $\mathcal{A}$ be a centrally closed prime algebra such that $\dim(T(\mathcal{A})) > 1$ for all nonzero T in the multiplication algebra $M(\mathcal{A})$ of $\mathcal{A}$. Then, there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}$ and $\{T_n\}_{n\in\mathbb{N}}$ in $M(\mathcal{A})$ such that $T_n T_{n-1} \cdots T_1 b_n \neq 0$ and $T_{n+1} \cdots T_1 b_n = 0$ for all $n \in \mathbb{N}$.*

Lemma 3.2 ([3]). *Every H^*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, and these are topologically simple H^*-algebra.*

Lemma 3.3 ([2]). *Every topologically simple H^*-algebra is centrally closed prime algebra.*

Lemma 3.4 ([9]). *Let $\mathcal{A}$ be an algebra, and assume the existence of a non-degenerate symmetric associative bilinear form $< . , . >$ on $\mathcal{A}$. Then we have*

(i) *there exist a unique linear algebra involution # on the multiplication algebra $M(\mathcal{A})$ of $\mathcal{A}$ satisfying $L_d^{\#} = R_d$ and $R_d^{\#} = L_d$ for all $d \in \mathcal{A}$;*

(ii) *for all $x, y \in \mathcal{A}$ and $T \in M(\mathcal{A})$. The equality $\langle Tx, y\rangle = \langle x, T^{\#} y\rangle$ holds.*

Lemma 3.5. *Let D be a g-c-derivation on topologically simple complete normed algebra $\mathcal{A}$ and suppose that there exists a nonzero $T \in M(\mathcal{A})$ with finite dimensional range satisfying $DT = TD$. Then D is continuous.*

Proof. Let $x \in \mathcal{S}(D)$, then there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that $\lim a_n = 0$ and $\lim Da_n = x$. Now, since TD is continuous and $\lim a_n = 0$, then $\lim TD(a_n) = 0$, hence $T(x) = 0$. Therefore, $T(\mathcal{S}(D)) = \{0\}$. But $\mathcal{S}(D)$ is closed ideal of $\mathcal{A}$, then $T\overline{(\mathcal{S}(D))} = \{0\}$. Since $\mathcal{A}$ topologically simple, we have $T(\mathcal{A}) = \{0\}$. This is a contradiction because T is nonzero. Then $\mathcal{S}(D) = \{0\}$. Thus D is continuous by the closed-graph theorem. □

A well-known result is due to Villena [11, Theorem 4], which states that: If $\mathcal{A}$ is H^*-algebra with zero annihilator, then any derivation on $\mathcal{A}$ is continuous. We present this result in a more general setting and we will use the nonassociative Jewell-Sinclair theorem in our proof, as we will see that in the following theorem:

Theorem B. *Every g-c-derivation on H^*-algebra $\mathcal{A}$ with zero annihilator is continuous.*

Proof. Let D be a g-c-derivation on H^*-algebra $\mathcal{A}$ with zero annihilator. At first, we assume that $\mathcal{A}$ is topologically simple. Applying Lemma 3.3, we have $\mathcal{A}$ is a centrally closed prime algebra. Now, $M(\mathcal{A})$ satisfying one of the following cases:

(i) There exist an element T in $M(\mathcal{A})$ such that T has finite dimensional range and $DT = TD$.

(ii) Every element in $M(\mathcal{A})$ has infinite dimensional range or $DT \neq TD$ for all T in $M(\mathcal{A})$.

First case. If (i) is true, then by using Lemma 3.5, D is continuous.

Second case. If (ii) is true, since every element in $M(\mathcal{A})$ has infinite dimensional range, it follows that from Lemma 3.1, there exists a sequence $\{C_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ and $\{T_n\}_{n\in\mathbb{N}} \subseteq M(\mathcal{A})$ such that for all $n \in \mathbb{N}$,

$$T_{n+1}T_n \cdots T_1 C_n = 0,$$

$$T_n T_{n-1} \cdots T_1 C_n \neq 0.$$

Taking into account that every topologically simple H^*-algebra contains a non-degenerate symmetric associative bilinear continuous form $< . , . >$ with an algebra involution # on $M(\mathcal{A})$ satisfying:

$L_b^{\#} = R_b$, $R_b^{\#} = L_b$ for all $b \in \mathcal{A}$. Now, if we suppose that $n \in \mathbb{N}$ is a positive integer number satisfying:

$\overline{T_1^{\#} \cdots T_n^{\#}(\mathcal{A})} = \overline{T_1^{\#} \cdots T_{n+1}^{\#}(\mathcal{A})}$, then we have a contradiction. Indeed,

$$0 = \langle \mathcal{A}, T_{n+1} \cdots T_1(C_n)\rangle = \langle \mathcal{A}, ((T_{n+1} \cdots T_1)^{\#})^{\#}(C_n)\rangle = \langle (T_{n+1} \cdots T_1)^{\#}(\mathcal{A}), C_n\rangle$$

$$= \langle T_1^{\#} \cdots T_{n+1}^{\#}(\mathcal{A}), C_n\rangle = \langle \overline{T_1^{\#} \cdots T_{n+1}^{\#}(\mathcal{A})}, C_n\rangle = \langle \overline{T_1^{\#} \cdots T_n^{\#}(\mathcal{A})}, C_n\rangle$$

$$= \langle T_1^{\#} \cdots T_n^{\#}(\mathcal{A}), C_n\rangle = \langle \mathcal{A}, ((T_n \cdots T_1)^{\#})^{\#}(C_n)\rangle = \langle \mathcal{A}, T_n \cdots T_1(C_n)\rangle.$$

Therefore, for every $n \in \mathbb{N}$, there exists a sequence $\left\{\overline{T_1^{\#} \cdots T_n^{\#}(\mathcal{A})}\right\}_{n\in\mathbb{N}}$ of closed right ideals of $\mathcal{A}$ constantly decreasing. Since $\mathcal{A}$ is semiprime, applying (Theorem A) we get D is continuous. In order to obtain the general case of the proof, assume that $\mathcal{A}$ has zero annihilator. Now, let M be a minimal closed ideal of $\mathcal{A}$. If the inclusion $D(M) \subseteq M$ is not true, then there exists a nonzero minimal closed J of $\mathcal{A}$ such that $D(M) \subseteq J$, and $J \cap M = \{0\}$. This is a contradiction because $\mathcal{A}$ is semiprime. Then the inclusion must be $D(M) \subseteq M$, or $D(M) \subseteq J \cap M \subseteq M$. Since M is topologically simple by Lemma 3.2, it follows from the first part of the proof that D is continuous on M. Let $a \in \mathcal{S}(D)$, then there exists a sequence $\{a_n\}$ in $\mathcal{A}$ such that $\lim a_n = 0$ and $\lim Da_n = a$. Now, for all $y \in M$, we have $\lim ya_n = 0$, since M is an ideal, then $ya_n \in M$. Since D is continuous on M, it follows that $\lim D(ya_n) = 0$. Now, $D(ya_n) = D(y)g(a_n) + yD(a_n)$, then $\lim D(ya_n) = ya$. Therefore $ya = 0$. Similarly, $ay = 0$. For all $a \in \mathcal{S}(D)$ and $y \in M$, then $0 = M\mathcal{S}(D) = \mathcal{S}(D)M$, for each minimal closed ideal M of $\mathcal{A}$. Therefore $0 = \mathcal{A}\mathcal{S}(D) = \mathcal{S}(D)\mathcal{A}$, which implies that $\mathcal{S}(D) \subseteq \mathrm{Ann}(\mathcal{A}) = \{0\}$. Thus D is continuous by the closed-graph theorem. □

References

[1] F. F. Bonsall and J. Duncan, Complete Normed Algebras, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[2] M. Cabrera and A. Rodriguez, Extended centroid and central closure of semiprime normed algebras: A first approach, Comm. Algebra 18 (1990), 2293-2326.

[3] J. A. Cuenca and A. Rodriguez, Structure theory for noncommutative Jordan H^*-algebra, J. Algebra 106 (1987), 1-14.

[4] T. S. Erickson, W. S. Martindale III and J. M. Osborn, Prime nonassociative algebras, Pacific J. Math. 60 (1975), 49-63.

[5] N. P. Jewell and A. M. Sinclair, Epimorphosis and derivations on $L_1(0, 1)$ are continuous, Bull. London Math. Soc. 8 (1979), 135-139.

[6] A. A. Mohammed, Y. Abdulljabar and N. A. Abdulraziq, Jewell-Sinclair theorem and the automatic continuity of a derivation, J. Edu. & Sci. 23(3) (2010), 63-69.

[7] A. A. Mohammed and S. M. Ali, On Villena's theorem of automatic continuity of essentially defined derivations on semisimple Banach algebras, Int. Journal of Math. Analysis 7(59) (2013), 2931-2939.

[8] T. W. Palmar, Banach Algebras and the General Theory of *-Algebras, I, Cambridge Uni. Press, 1994.

[9] A. Rodriguez, Continuity of densely valued homeomorphisms into H^*-algebra, Quart. J. Math. Oxford (2)46 (1995), 107-118.

[10] T. M. Sinclair, Automatic Continuity of Linear Operators, Cambridge Uni. Press, 1976.

[11] A. R. Villena, Continuity of derivations on H^*-algebra, Proc. Amer. Soc. 122(3) (1994), 821-826.

■